Journal of Statistical Physics, Vol. 119, Nos. 5/6, June 2005 (© 2005)
DOI: 10.1007/s10955-005-5106-6

Two Level Systems Driven by a Stochastic Perturbation

Marco Maioli[1] **and Andrea Sacchetti**[1]

Received October 28, 2004; accepted February 2, 2005

Here we consider a two level system driven by an external harmonic field whose amplitude is perturbed by a white noise term. In the limit of small splitting, dynamical localization, i.e. coherent destruction of tunneling, is proved for times of the order of $1/\epsilon$, where ϵ is the two-level splitting. The same type of localization is proved if the driving field is simply the white noise.

KEY WORDS: Two level quantum systems; dynamical localization; random processes; estimates of covariance; 81-xx; 82A31; 60H10

1. INTRODUCTION

Suppression of quantum coherence – or destruction of tunnelling or, also, dynamical localization – is a remarkable phenomenon in variuos models of two level systems submitted to a suitable external periodic driving force: it is often found for resonant values of the parameters of the driving term (see e.g. refs. 1, 5, and 7 and therein references).

The role of a driving stochastic perturbation is currently investigated by various methods. For example the sum of a periodic term and a random perturbation is considered in ref. 6, and the resulting driven system is studied by a Wiener–Hermite expansion and by numerical methods. In ref. 9, the authors consider an open two-level system driven by a circularly polarized field with Hamiltonian

$$H(t) = \epsilon\sigma_1 + \alpha\xi(t)\sigma_3 + [(V_\circ/2) + \gamma\eta(t)][\sigma^+ \exp(i\omega t) + \sigma^- \exp(-i\omega t)],$$

[1]Department of Mathematics, University of Modena and Reggio Emilia, Via Campi 213/B, 41100 Modena, Italy; e-mail: maioli.marco@unimore.it, sacchetti.andrea@unimore.it

where ϵ is the energy difference relative to the isolated two-level system, σ denote the Pauli matrices (3), $\sigma^{\pm} = \sigma_1 \pm \sigma_2$, and $\xi(t), \eta(t)$ are random functions of Gaussian white noise. The analysis in ref. 9 is performed by averaging techniques and deals with the probability of the system occupying one of the two localized states. The time evolution of such quantity numerically appears convergent to $1/2$, i.e. the effect of noise added to a circularly polarized field is to destabilize the localized state. To sum up, both in refs 6 and 9 no localization results appear in the large time regime. In the case of a degenerate energy level, split by an external monochromatic field, there are results of restoration of the degenerate state by an appropriate noise;[11] while the occurence of quantum stochastic resonances is studied in driven biased spin-boson systems.[4]

In our model we consider the case of a periodic driving external field with amplitude modulated by means of a white noise term (see Eq. (12) below for details). In such a case we can conclude (see Theorem B) that dynamical localization is always induced by the presence of the noise in the sense recalled in the following section and for times of the order $1/\epsilon$, where ϵ is the two-level splitting. So it appears as an opposite result with respect to ref. 9, where a different combination of white noise with the external periodic field was presented. By similar methods we show the occurrence of dynamical localization (see Theorem D) if the driving field is simply a white noise term, i.e. for a model which was studied by averaging methods in ref. 2. As suggested below (Remark 4) our results lead to distinguish the asymptotics of the system as $t \to \infty$ from the localizing role of white noise for times of the order $O(\epsilon^{-1})$, where ϵ is the splitting of the energy levels.

Our model would be useful in order to describe the effect of the environment on a single symmetric quantum system (for instance, with a double well potential) by means of a random external driven field. We show the existence of localized states in the limit of small splitting ϵ. In such a model the unperturbed beating motion has period of the order ϵ^{-1}; hence it is usual to rescale the time by considering the "slow time" $\tau = t\epsilon$. As a result, the appearance of classical behaviour in the limit of small splitting follows: that is the wave function will stay localized in one of the two wells for any fixed "time" τ.

Our main tools are the use of a general criterion[8] for dynamical localization and estimates on covariance related to the rate of convergence of random processes.

2. NOTATION AND PRELIMINARY RESULTS

Driven two-level systems in quantum mechanics can appear when a symmetric potential and an external time-dependent field is considered.

See, for instance, ref. 5, where the time-dependent Schrödinger equation takes the form

$$i\frac{\partial}{\partial t}u(x,t) = -\frac{\partial^2}{\partial x^2}u(x,t) + [x^4 - \beta x^2]u(x,t) + Sx\sin(\omega t)u(x,t), \qquad S,\beta > 0. \tag{1}$$

In general, if the autonomous Hamiltonian has two parity-even and -odd eigenstates $v_\pm$ with eigenvalues $E_\pm$, the restriction of Eq. (1) to the bi-dimensional space spanned by the two eigenvectors $v_\pm$ is usually called *two-level system* and, in a suitable base, it takes the following form

$$i\dot{\phi} = H_1\phi, \qquad H_1 = \epsilon\sigma_1 + \eta f(\omega t)\sigma_3, \qquad \phi(0) = \phi^0, \tag{2}$$

where $\epsilon = 1/2|E_+ - E_-| > 0$, η is a real-valued parameter directly proportional to the field's strength, $\dot{\phi}$ denotes the derivative of ϕ with respect to the time t,

$$\phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix},$$

while ω is the driving frequency, $f(t)$ is a given periodic function with period 2π and $\sigma_{1,2,3}$ are the Pauli's matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

If the external field is absent, that is $\eta = 0$, then Eq. (2) takes the form

$$i\dot{\phi} = \epsilon\sigma_1\phi$$

with solution given by

$$\phi_1(t) = \phi_1(0)\cos\epsilon t - \phi_2(0)\sin\epsilon t,$$
$$\phi_2(t) = \phi_1(0)\sin\epsilon t + \phi_2(0)\cos\epsilon t.$$

Hence $\phi(t)$ is a periodic function with period $2\pi/\epsilon$ and the wave-function $u(x,t)$ shows a beating motion between the two wells.

When restoring the driving field it is useful to write the original Eq. (2) in a different form by the transformation

$$\psi = e^{i\alpha\sigma_3}\phi,$$

where

$$\alpha(t) = \int_0^t \eta f(\omega\ s) ds. \tag{4}$$

Then Eq. (2) takes the form

$$i\dot{\psi} = H_2 \psi, \qquad H_2 = \epsilon e^{i\alpha\sigma_3} \sigma_1 e^{-i\alpha\sigma_3}, \tag{5}$$

with the same initial condition $\psi(0) = \psi^0 = \phi^0$. By means of the averaging theorem[10] in the limit of small beating frequency, that is $\epsilon \ll \omega$, and for times of the order of the beating period T we can approximate the solution of Eq. (5) by the solution of the average system given by

$$i\dot{\hat{\psi}} = \hat{H}_2 \hat{\psi}, \qquad \hat{H}_2 = \epsilon \hat{I} \sigma_1, \tag{6}$$

where

$$\hat{I} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} e^{2i\alpha(t)} dt = \frac{1}{2\pi} \int_0^{2\pi} e^{2i\chi \int_0^t f(q)dq} dt, \quad \chi = \frac{\eta}{\omega}. \tag{7}$$

That is, the unperturbed solution $\psi(t)$ is approximated by means of the solution $\hat{\psi}$ related to the averaged Eq. (6) for any time of the order $1/\epsilon$: for any $\delta > 0$ there exists $\epsilon_0 > 0$ such that for any ϵ, $0 < \epsilon < \epsilon_0$, then

$$|\psi(t) - \hat{\psi}(t)| < \delta, \quad \forall t \in [0, T]. \tag{8}$$

In particular, if $\hat{I} = 0$ then the wavefunction ψ is found to be nearly "frozen" in its initial configuration (for times of the order of T): we have the so-called dynamical localization effect according to the following definition:

Definition. Let ψ be the solution of equation (5) with initial condition ψ^0, let $T = 2\pi/\epsilon$ be the unperturbed beating period. The coherent destruction of the tunneling, also called dynamical localization effect, means that for any $\nu > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ then

$$|\psi(t) - \psi^0| < \nu, \quad \forall t \in [0, T]. \tag{9}$$

This analysis can be extended to non-periodic external fields by means of the following criterion.

Criterion A. [8]. *Let $f(t)$ be a given integrable function (not necessarily periodic). The driven two-level system*

$$i\dot{\phi} = H_1\phi, \qquad H_1 = \epsilon\sigma_1 + f(t)\sigma_3, \qquad \phi(0) = \phi^\circ, \tag{10}$$

satisfies dynamical localization if

$$\hat{I} \equiv \lim_{d\to\infty} \frac{1}{d}\int_0^d e^{2i\alpha(t)}dt = 0 \tag{11}$$

where $\alpha(t) = \int_0^t f(s)ds$.

Such a condition is necessary, too, in the case of periodic f.

Remark 1. In fact in ref. 8, just for sake of definiteness, piecewise continuity of f was assumed. However the above criterion still works for any integrable function.

In the following section we ask whether a suitable insertion of white noise, denoted $\xi(t)$, in a harmonic driving term

$$f(t) = \eta[1 + \lambda\ \xi(t)]\cos(\omega t), \quad t \geqslant 0,\ \lambda > 0, \tag{12}$$

implies dynamical localization in the corresponding system (10).

As a result we obtain dynamical localization (Theorems B, and C) of the system (10) for any strength λ of the white noise and for all parameters η, ω.

3. WHITE NOISE INSERTED AND MAIN RESULT

First, for sake of simplicity, the notation with subscript "t" will be adopted for the processes dealing with (12): ξ_t, f_t and α_t in place of $\xi(t)$, $f(t)$ and $\alpha(t) = \int_0^t f(s)ds$. Sometimes the mention of λ, but not of η, ω, will be made.

We denote by $\{\xi_t\}_{t\geqslant 0}$ the white noise process, formally specified by the relation

$$\int_a^b g(t)\ \xi_t dt = \int_a^b g(t)\ dW_t \quad \forall g, \forall a, b \geqslant 0(). \tag{13}$$

Here $\{W_t\}_{t\geqslant 0}$ is the Wiener process, and on the right hand side denotes the Ito integral of arbitrary g's in a suitable class of non-random functions.[3]

The delta-autocorrelation property is the main property ([3], p. 80) of this stationary process: $E[\xi_t \xi_s] = \delta(t-s)$ for any $t, s > 0$. In principle it is shared by other processes, still named "white noise", and it corresponds to a uniform assignment of probability to all frequences, with some analogy with "white" light. Here we recall the following further properties, which are inherited from the connection (13) with W_t ([3], p. 103): for any non-random function $b(t)$,

$$\forall s,\, t>0, \quad \int_s^t b(t')\ \xi_{t'} dt' \text{ is Gaussian, with } \quad E\left[\int_s^t b(t')\ \xi_{t'} dt'\right] = 0,$$

$$E\left[\int_{t_\circ}^t b(t')\ \xi_{t'} dt' \int_{t_\circ}^s b(t')\ \xi_{t'} dt'\right] = \int_{t\circ}^{\min(t,s)} [b(t')]^2 dt', \quad \forall t_\circ \leqslant s, t$$

$$\mathrm{Var}\left[\int_s^t b(t')\ \xi_{t'} dt'\right] = \left|\int_s^t [b(t')]^2 dt'\right|, \quad \forall s,\, t. \tag{14}$$

In this frame we want to prove:

Theorem B. The two level system

$$i\dot{\phi}(t) = \{\epsilon\sigma_1 + f_t(\lambda)\sigma_3\}\phi(t), \quad \phi(0) = \phi_\circ \tag{15}$$

with driving term

$$f_t(\lambda) = \eta[1 + \lambda\ \xi_t]\cos(\omega t), \quad t \geqslant 0 \tag{16}$$

satisfies dynamical localization with probability one, for any strength $\lambda > 0$ of the white noise ξ_t, and for all parameters $\eta, \omega > 0$.

Remark 2. As proved in ref. 7, for $\lambda = 0$ there is dynamical localization only for particular resonant parametrs η, ω. It seems rather surprising that, even for small $\lambda > 0$, dynamical localization takes place for all η, ω. The explanation of this apparent paradox is simple: in the definition of dynamical localization, i.e. formula (9), a very small $\epsilon_\circ$ has to be taken when λ is small. On the other hand, such explanation agrees with the fact that the estimates below are $\lambda-$dependent in a singular way as $\lambda \to 0$. For example in (39) $E[I(d, \lambda)]$ vanishes as $d \to \infty$ with a factor λ^{-2}; and similarly in (27) and (28) one finds $\delta = O(\lambda^2)$ and $c = O(\lambda^{-2})$ as $\lambda \to 0$.

Proof of Theorem B. In order to apply Criterion A, we need a good knowledge of the processes α_t, $\exp(2i\alpha_t)$, where α_t is obtained by integration of (16). That is the content of Lemmas 1 and 2.

Lemma 1. The stochastic process

$$\alpha_t(\lambda) := \int_0^t \eta\cos(\omega s)ds + \lambda\int_0^t \eta\cos(\omega s)dW_s \tag{17}$$

is Gaussian with mean and covariance

$$E[\alpha_t(\lambda)] = \int_0^t \eta\cos(\omega t')dt', \qquad \mathrm{Cov}[\alpha_t, \alpha_s] = \lambda^2\eta^2\int_0^{min[t,s]} \cos^2(\omega t')dt'. \tag{18}$$

Proof. An application of properties (14).

Lemma 2. The stochastic process $\{\exp(2i\alpha_t)\}_{t\geqslant 0}$ has mean and covariance

$$E[\exp(2i\alpha_t)] = \exp\left[-2\int_0^t \lambda^2\eta^2\cos^2(\omega t')dt'\right]\exp\left[2i\int_0^t \eta\cos(\omega t')dt'\right]. \tag{19}$$

$$\mathrm{Cov}\left[e^{2i\alpha_t}, e^{2i\alpha_s}\right] = \exp\left(+2i\int_s^t \eta\cos(\omega t')dt'\right)\cdot\left\{\exp\left(-2\left|\int_s^t \lambda^2\eta^2\cos^2(\omega t')dt'\right|\right)\right.$$
$$\left. -\exp\left(-2\lambda^2\eta^2\int_0^t \cos^2(\omega t')dt' - 2\lambda^2\eta^2\int_0^s \cos^2(\omega t')dt'\right)\right\}. \tag{20}$$

Proof. As it is known, any Gaussian variable Z with zero mean satisfies: $E[\ \exp Z\ ] = \exp(1/2E[Z^2])$. So, by analytic continuation

$$E[\ \exp(cZ)\ ] = \exp(1/2c^2E[Z^2]), \quad \forall c\in C.$$

Choosing $c = 2i$, $\ Z = \alpha_t - E[\alpha_t]$,

$$\begin{aligned} E[\exp(2i\alpha_t)] &= E[\ \exp(2i\alpha_t - 2iE[\alpha_t])\ ]\cdot\exp(2iE[\alpha_t]) \\ &= \exp(1/2E[4i^2(\alpha_t - E[\alpha_t])^2]\cdot\exp(2iE[\alpha_t]). \end{aligned} \tag{21}$$

Now, by virtue of (18),

$$E[\alpha_t] = \int_0^t \eta \cos(\omega s) ds, \qquad Var[\alpha_t] = \int_0^t \lambda^2 \eta^2 \cos^2(\omega s) ds. \tag{22}$$

Thus (19) holds.

As for (20), we recall the covariance of complex random variables: X, Y with means μ_X, μ_Y: $\mathrm{Cov}(X, Y) = E[X\bar{Y}] - \mu_X \bar{\mu}_Y$. Thus setting $K(t, s) = \mathrm{Cov}[\exp(2i\alpha_t), \exp(2i\alpha_s)]$, such covariance is equal to

$$\begin{aligned} K(t,s) &= E[\exp(2i(\alpha_t - \alpha_s))] - E[\exp(2i\alpha_t)] \cdot E[\exp(-2i\alpha_s)] \\ &= E[e^{2i(\alpha_t - \alpha_s) - 2iE[\alpha_t - \alpha_s]}] \cdot e^{2iE[\alpha_t - \alpha_s]} - E[e^{2i\alpha_t}]\ E[e^{-2i\alpha_s}] \end{aligned} \tag{23}$$

Now using (19) and the identity on Gaussians with mean zero, (23) can be written

$$\left\{ \exp(-2E[(\alpha_t - \alpha_s - E[\alpha_t - \alpha_s])^2]) - \exp\left(-2\int_0^t \lambda^2 \eta^2 \cos^2 \omega t' dt' \right.\right. \\ \left.\left. -2\int_0^s \lambda^2 \eta^2 \cos^2 \omega t' dt'\right)\right\} \cdot \exp\left(2i \int_s^t \eta \cos \omega t' dt'\right).$$

The first exponent is equal to $-2\ E[(\alpha_t - \alpha_s)^2] + 2\ (E[\alpha_t - \alpha_s])^2$. Now, for any $b(t)$, $\int_s^t b(t') dW_{t'}$ has mean zero and variance $|\int_s^t b(t')^2 dt'|$, so we find:

$$\begin{aligned} E[(\alpha_t - \alpha_s)^2] &= E\left[\left\{\int_s^t \eta \cos(\omega t') dt' + \int_s^t \lambda \eta \cos(\omega t') dW_{t'}\right\}^2\right] \\ &= E\left[\left\{\int_s^t \eta \cos(\omega' t') dt'\right\}^2 + \left\{\int_s^t \lambda \eta \cos(\omega t') dW_{t'}\right\}^2\right] \\ &\quad + E\left[2\left\{\int_s^t \eta \cos(\omega' t') dt'\right\} \cdot \left\{\int_s^t \lambda \eta \cos(\omega t') dW_{t'}\right\}\right] \\ &= \left\{\int_s^t \eta \cos(\omega t') dt'\right\}^2 + \left|\int_s^t \lambda^2 \eta^2 \cos^2(\omega t') dt'\right|. \end{aligned} \tag{24}$$

Then there is a cancellation $(-2\{\int_s^t\}^2 + 2\{\int_s^t\}^2)$ and the covariance (23) is written finally:

$$\left\{ \exp\left(-2\ \left|\int_s^t \lambda^2 \eta^2 \cos^2(\omega t') dt'\right|\right) - \exp\left(-2\lambda^2\eta^2 \int_0^t \cos^2(\omega t') dt' \right.\right. \\ \left.\left. -2\lambda^2\eta^2 \int_0^s \cos^2(\omega t') dt'\right)\right\} \cdot \exp\left(+2i \int_s^t \eta \cos(\omega t') dt'\right). \tag{25}$$

Remark 3. In particular the variance is positive $\forall t>0$

$$\mathrm{Var}(\exp[2i\alpha_t]) = 1 - \exp[-4\int_0^t \lambda^2\eta^2\cos^2(\omega t')dt'] \tag{26}$$

with asymptotics $O(t)$ as $t\to 0$, $O(1)$ as $t\to\infty$.

Lemma 3. There exists $\delta>0$ such that

$$|\mathrm{Cov}[\exp 2i\alpha_t,\ \exp 2i\alpha_s]\ | \leqslant \exp(-\delta|t-s|) + \exp(-\delta(t+s)), \quad \forall t,\ s\geqslant 0. \tag{27}$$

Proof. In the explicit expression (20), we need a lower bound of the integral:

$$\left|\int_s^t \cos^2(\omega t')dt'\right| = \left|\frac{1}{2\omega}(t-s) - \frac{1}{4\omega}\{\sin(2\omega t) - \sin(2\omega s)\}\right| \geqslant \delta'|t-s|.$$

A similar lower bound for the other integral in (20) is given by $\delta' t + \delta' s$. By such bounds of exponents in (20), the estimate (27) is obtained.

One can note the non-uniformity of such δ's with respect to λ: indeed $\delta = \lambda^2\eta^2\delta'$. Lemma 3 are useful to study the (limit of) integrals appearing in Criterion A:

Lemma 4. Let $I(d,\lambda) = d^{-1}\int_0^d \exp[2i\alpha_t(\lambda)]dt$ for all $d,\lambda>0$. Then its variance satisfies

$$\exists c>0: \quad \forall d>0,\ \ 0\leqslant \mathrm{Var}[I(d,\lambda)] \leqslant \frac{c}{d}. \tag{28}$$

Proof. Fix $\lambda>0$. In this proof let us use the notation $\alpha(t)$ in place of $\alpha_t(\lambda)$. For $n\in N$ and $k=1,\dots,n$ consider the points $t_k = kd/n$ in the interval $[0,d]$ and the associated Riemann sums:

$$I(d,\lambda) = \frac{1}{d}\sum_{k=1}^{n}\frac{d}{n}\exp 2i\alpha\left(\frac{kd}{n}\right) + R_n(d,\lambda). \tag{29}$$

Here $\lim_n\ R_n = 0$ almost everywhere, since the integrand is a.e. continuous (α_t has the same regularity of the Wiener process W_t). Since $R_n\to 0$ a.e. in a space with finite measure, by Egoroff's theorem it is convergent almost uniformly and, a fortiori, in measure. Furthermore R_n is bounded since

$$|R_n| \leq \ |I(d,\lambda)| + \frac{1}{d}\left|\sum_{k=1}^{n}\frac{d}{n}\exp 2i\alpha\left(\frac{kd}{n}\right)\right| \ \leqslant\ 1+1. \tag{30}$$

Finally the boundedness and the convergence in measure imply convergence in L^{∞} of the probability space. In particular $R_n \to 0$ in L^2 and the inequality

$$\| I(d,\lambda) \|_2 \leqslant \left\| \frac{1}{d}\sum_{k=1}^{n}\frac{d}{n}\exp 2i\alpha\left(\frac{kd}{n}\right)\right\|_2 + \| R_n \|_2, \tag{31}$$

reduces the estimate of $I(d,\lambda)$ to an estimate of the Riemann sum. We have:

$$\begin{aligned}
\mathrm{Var}[I(d,\lambda)] &\leqslant \cdot \mathrm{Var}\left[\frac{1}{d}\sum_{k=1}^{n}\frac{d}{n}\exp 2i\alpha\left(\frac{kd}{n}\right)\right] \\
&= \frac{1}{n^2}\sum_{j,k}\mathrm{Cov}\left[\exp 2i\alpha\left(\frac{jd}{n}\right), \exp 2i\alpha\left(\frac{kd}{n}\right)\right] \\
&\leqslant \frac{A}{n^2}\cdot\left\{\sum_{j,k}\exp\left(-\delta|j-k|\frac{d}{n}\right) + \sum_{j,k}\exp\left(-\delta(j+k)\frac{d}{n}\right)\right\},
\end{aligned} \tag{32}$$

where the inequality (27) has been used. Let us consider each of the two sums in (32): the first one is not greater than

$$\frac{2}{n^2}\sum_{j\leqslant k}\exp\left(-\delta|j-k|\frac{d}{n}\right) = \frac{2}{n^2}\left\{n + (n-1)e^{-\delta d/n} + \cdots + 2e^{-(n-2)\delta d/n} \right. \\ \left. + e^{-(n-1)\delta d/n}\right\} \tag{33}$$

Now, setting $x \equiv y^{-1} := e^{-\delta d/n}$, we use the identity

$$\begin{aligned}
\sum_{r=1}^{n} r x^{n-r} \ &\equiv y^{-n}\sum_{r=1}^{n} r y^{r} \ \equiv y^{-n+1}\frac{d}{dy}\sum_{1}^{n} y^{r} \\
&= y^{-n+1}\frac{d}{dy}\left\{\frac{y^{n+1}-1}{y-1} - 1\right\} \equiv \frac{y}{(y-1)^2}\left\{ny - (n+1) + y^{-n}\right\}
\end{aligned} \tag{34}$$

which yields

$$\frac{2}{n^2}\sum_{j\leqslant k}\exp\left(-\delta|j-k|\frac{d}{n}\right) \sim \frac{2}{\delta d} \text{ as } n\to\infty. \tag{35}$$

The second sum in (32) is less than the first one, so we obtain $\mathrm{Var}[I(d,\lambda)] \leqslant c/d$ as stated.

Lemma 5. For any $\lambda>0$

$$\exists \lim_{d\to\infty}\{I(d,\lambda)-E[I(d,\lambda)]\}=0, \tag{36}$$

almost surely.

Proof. By (19) the mean value $E[I(d,\lambda)]=\frac{1}{d}\int_0^d E[\exp 2i\alpha_t(\epsilon)]dt$ is finite $\forall d>0$. For $d\in N$, $\gamma>0$, using Chebyshev inequality and the estimate of Lemma 4 we check:

$$P\{|I(d,\lambda)-E[I(d,\lambda)]\ |>\gamma\}\leqslant\frac{\mathrm{Var}\{I(d,\epsilon)\}}{\gamma^2}\leqslant\frac{c}{\gamma^2 d}, \quad \forall d\in N,\ \forall\gamma>0. \tag{37}$$

Thus the difference $I(d,\lambda)-E[I(d,\lambda)]$ tends to 0 in probability. If the inequality (37) is considered with d replaced by d^2, by the first Borel–Cantelli lemma the events $|I(d^2,\lambda)-E[I(d^2,\lambda)]|>\gamma$ have probability zero from some $d=d(\gamma)$ on. Since γ is arbitrary, the subsequence $I(d^2,\lambda)-E[I(d^2,\lambda)]$ is convergent almost surely as $d\to\infty$. But such convergence extends to the whole sequence: indeed setting $d=m^2+\beta(d)$, where m is the integer part of $\sqrt{d}$, we notice that $\beta\leqslant 2m+1$: so

$$|I(d,\lambda)-E[I(d,\lambda)]|=\left|\frac{1}{d}\int_0^d\{e^{2i\alpha_t}-E[e^{2i\alpha_t}]\}dt\ \right|$$

$$\leqslant\left|\frac{1}{m^2+\beta}\int_0^{m^2}...dt\right| + \frac{1}{m^2+\beta}\int_{m^2}^{m^2+\beta}|...|dt,$$

where the first term tends to zero as the above subsequence, while the second is bounded by $const.\cdot\beta(d)/[m^2+\beta(d)]\ \leqslant(2m+1)/m^2\to 0$ as $m\to\infty$. This proves the lemma.

The above results allow to prove Theorem B. Indeed, by Criterion A it is enough that

$$\lim_{d\to\infty} I(d,\lambda)=0 \quad a.s. \tag{38}$$

Now the mean value of $I(d,\lambda)$ vanishes as $d\to\infty$: indeed by (19) there is $\delta'>0$ such that

$$\begin{aligned}|E[I(d,\lambda)]\,| &= \frac{1}{d}\int_0^d \exp\left[-\int_0^t \lambda^2\eta^2\cos^2(\omega t')dt'\right]dt \leqslant \frac{1}{d}\int_0^d \exp[-\delta'\lambda^2 t]dt \\ &= \frac{1}{d}[e^{-\lambda^2\delta' t}/(-\lambda^2\delta')]_0^d \sim \frac{1}{d\lambda^2\delta'} \quad \text{as } d\to\infty. \end{aligned} \tag{39}$$

Then, by virtue of Lemma 5, $I(d,\lambda)$ tends to zero almost surely as $d\to\infty$. Thus Theorem B is completely proved.

There is a wider version of the Theorem, with the following statement.

Theorem C. Let $g(t)$ be a real-analytic periodic function for which the only zeroes are simple. Then the two level system (15) with driving term

$$f_t(\lambda)=\eta[1+\lambda\ \xi_t]g(\omega t), \quad t\geqslant 0 \tag{40}$$

satisfies dynamical localization with probability one, for any strength $\lambda>0$ of the white noise ξ_t, and for all parameters $\eta,\omega>0$.

Proof. If $\cos(t)$ is replaced by $g(t)$, the above arguments work without modifications if the assumption of simple zeroes is made. Indeed in such case the lower bound used in Lemma 3, $|\int_s^t [g(\omega t')]^2 dt'|\geqslant\delta|t-s|, \forall s,t>0$, is guaranteed, and the final statement is proved.

Finally we notice that localization is similarly stabilized by white noise alone:

Theorem D. Let $\lambda>0$ and $f(t)=\lambda\xi_t$. Then the two-level system (15) satisfies dynamical localization in the same sense of the above theorems.

Proof. It suffices to adapt the arguments outlined in the proof of Theorem B. Indeed $\alpha_t=\lambda W_t$, so that

$$E[\exp(2i\alpha_t)]=\exp(-2\lambda^2 t),$$

$$\mathrm{Cov}[e^{2i\alpha_t}, e^{2i\alpha_s}] = \exp(-2\lambda^2|t-s|) - \exp[-2\lambda^2 t - 2\lambda^2 s].$$

From this point on, the same arguments of Lemmas 3–5 hold and the theorem is proved.

Remark 4. Is is interesting to compare Theorem D with the results of ref 2, where the same Hamiltonian $H = \epsilon\sigma_1 + \lambda\xi_t\sigma_3$ is studied by averaging methods. There two regimes are distinguished: when $\lambda^2 < 2|\epsilon|$ the systems jumps from one state to the other almost periodically with damping due to the white noise; when $\lambda^2 > 2|\epsilon|$ the system jumps randomly from one state to the other with probability rate vanishing like $2\epsilon^2/\lambda^2$ as $|\epsilon|/\lambda \to 0$. Now, by Theorem D, this last regime is specified: in fact the localization in the initial state is almost surely preserved for times of the order of $2\pi/\epsilon$ (see Criterion A), where ϵ is the splitting of the energy levels.

A similar interpretation should be associated to Theorems B and C: even if white noise perturbations of a driving periodic term do not prevent tunneling as $t \to +\infty$ (as suggested by Shao *et al.*[9]), yet they can stabilize the system in the initial state for long times if the two-level splitting is sufficiently small.

ACKNOWLEDGMENTS

This work is partially supported by the GNFM project "Comportamenti classici di sistemi quantistici."

REFERENCES

1. L. Allen, and J. H. Eberly, *Optical resonance and Two-level Atoms*. (Dover: New York, 1987).
2. Ph. Blanchard, G. Bolz, M. Cini, G. F. De Angelis, and M. Serva, Localization stabilized by noise, *J. Stat. Phys.* **75**:749–755 (1994).
3. C. W. Gardiner, *Handbook of Stochastic Methods* (Berlin: Springer, 1990).
4. M. Grifoni, and P. Hanggi, Coherent and incoherent quantum stochastic resonance, *Phys. Review Lett.* **76**:1611–1614 (1996).
5. M. Grifoni, and P. Hanggi, Driven quantum tunnelling, *Phys. Rep. A* **304**:229 (1998).
6. Y. Kayanuma, and K. Noba, Wiener–Hermite expansion formalism for the stochastic model of a driven quantum system. *Chem. Phys.* **268**:177–188 (2001).
7. D. Lodi, M. Maioli, and A. Sacchetti, Dynamical localization for two-level systems periodically driven, *J. Phys. A: Math. and General* **38**:L23–L30 (2005).
8. A. Sacchetti, Dynamical localization criterion for driven two-level systems, *J. Phys. A: Math. and General* **34**:10293–10306 (2001).
9. J. Shao, C. Zerbe, and P. Hanggi, Suppression of quantum coherence: noise effect, *Chem. Phys.* **235**:81–92 (1998).

10. J. A. Sanders, and F. Verhulst, *Averaging Methods in Nonlinear Dynamical Systems* (Berlin: Springer, 1985).
11. J. Zhang, and H. Xin, The effect of quantum noise on the energy degeneration in a dissipative quantum two-level system, *Chem. Phys. Lett.* **345**:282–286 (2001).